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Non-spherically symmetric transparent potentials for the three-dimensional Schrödinger equation

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Abstract
A class of transparent potentials (i.e., potentials with trivial scattering operator $S = 1$) for the three-dimensional Schrödinger equations is studied. We find the underlying group explaining the transparency phenomenon for these Hamiltonians.

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As is well known [1], a modified Pöschl–Teller potential hole has a vanishing reflection coefficient for all energies. It turns out that there is an infinite family of one-dimensional potentials for which the scattered wave has no reflected part [2–10]. Because of this property, these potentials are called reflectionless. Another remarkable property is that [11–15] the reflectionless potentials give rise to soliton solutions of nonlinear equations such as Korteweg–de Vries, sine-Gordon, nonlinear Schrödinger etc.

A method of constructing reflectionless potentials for the one-dimensional Schrödinger equation was first developed by Kay and Moses [2]. This method is based on the inverse scattering problem [16]. Other methods to find the reflectionless potentials are to use the Darboux techniques: the Darboux transformation, the supersymmetric quantum mechanics etc.

All the methods mentioned above apply to the one-dimensional Schrödinger equation as well as the radial Schrödinger equation with a central potential. In particular, a family of potentials yielding the scattering phase shift (of given angular momentum) equal to zero for all energies [8, 17, 18] has been constructed. Further it was shown in the frame of the inverse scattering problem that there exists a family of potentials for which all phase shifts are identically zero at fixed energy [16, 19, 20]. These potentials are called transparent [16]. The inverse scattering problem at fixed energy for the two-dimensional Schrödinger equation with potentials in a certain class that are transparent at a given energy was studied in [21].

In [22] we described a general method which allows pure algebraic calculation of S-matrices for the systems whose Hamiltonians are related to the Casimir operators C_i of some non-compact group G,

$$H = f(C_i), \tag{1}$$

or

$$H = f(C_i)|_{\mathfrak{H}}, \tag{2}$$

where $\mathfrak{H}$ are the subspaces occurring in the subgroup reductions. Namely, the S-matrices for the systems under consideration are associated with the intertwining operators A between the Weyl equivalent representations U^{χ} and $U^{\tilde{\chi}}$ of G,

$$S = A \tag{3}$$

or

$$S = A|_{\mathfrak{H}}, \tag{4}$$

respectively. (The representations U^{χ} and $U^{\tilde{\chi}}$ have the same Casimir eigenvalues. Such representations are called Weyl equivalent.) At this stage we note that the operator A is said to intertwine the representations U^{χ} and $U^{\tilde{\chi}}$ of the group G if the relation

$$AU^{\chi}(g) = U^{\tilde{\chi}}(g)A \qquad \text{for all} \quad g \in G \tag{5}$$

or

$$A\mathrm{d}U^{\chi}(b) = \mathrm{d}U^{\tilde{\chi}}(b)A \qquad \text{for all} \quad b \in \mathfrak{g} \tag{6}$$

holds, where $\mathrm{d}U^{\chi}$ and $\mathrm{d}U^{\tilde{\chi}}$ are the corresponding representations of the algebra $\mathfrak{g}$ of G. Equations (5) and (6) have much restriction power, determining the intertwining operator; one can easily evaluate, taking advantage of group-theoretic techniques, the S-matrix. This algebraic technique can be also used in the opposite procedure, i.e., given a scattering matrix related to an intertwining operator of some algebra $\mathfrak{g}$ as (3) or (4) to look for a representation of $\mathfrak{g}$ such that the operator H defined by (1) or (2) be a meaningful Hamiltonian. In particular, in this way we have constructed an infinite family of reflectionless potentials for the Schrödinger equation in one dimension [23].

In the present paper we are interested in applying the method to the three-dimensional Schrödinger equation. Our motivation for such a study can conveniently be understood by considering the following statement made by Matveev [24]: "The explicit multidimensional Schrödinger operator with trivial scattering operator and nontrivial potentials seems to be unknown." We show that the non-spherically symmetric potentials

$$V^{(1)}(\mathbf{x}) = a_0 \frac{n^2 - \frac{1}{4}}{2r^2 \sin^2\theta \sin^2\varphi} \tag{7}$$

and

$$V^{(2)}(\mathbf{x}) = a_0 \frac{n^2 - \frac{1}{4}}{2r^2 \cos^2\theta}, \tag{8}$$

with $n = 0, 1, 2, \ldots,$ are transparent for all energies. Here r, θ, φ are spherical coordinates, and a_0 is defined in terms of a mass parameter M by $a_0 = \hbar^2/M$. Henceforth we will set $M = \hbar = 1$.

The key to the group-theoretical construction of transparent potentials for the Schrödinger equation in three dimensions lies in the facts that (a) the N-dimensional Schrödinger equation with a null potential admit the Euclidean group in N dimensions $ISO(N)$ as a (maximal) symmetry group (or equivalently, the Hamiltonian H for a free particle in N dimensions is a multiple of the second-order Casimir operator C of $ISO(N)$) and (b) the null potential cause no scattering at all, i.e., $S = 1$. This suggests the following assumption that a certain transparent potentials for the Schrödinger equation in three dimensions must be related to $ISO(N)$, $N > 3$.

We show that the Hamiltonians corresponding to (7) and (8) are related to $ISO(4)$ in the sense that the following relation holds:

$$H^{(i)} = -\tfrac{1}{2}C\big|_{\mathfrak{H}^{(i)}}, \qquad i = 1, 2, \tag{9}$$

where C is the second-order Casimir operator of $ISO(4)$, while $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$ are subspaces occurring in the reductions $ISO(4) \supset SO(4) \supset SO(3) \supset SO(2)$ and $ISO(4) \supset SO(4) \supset SO(2) \times SO(2)$, respectively.

Let us start the discussion with the fact that [25, 26] the group $ISO(4)$, i.e., the semidirect product $T_4 \boxtimes SO(4)$ of the four-dimensional translation group T_4 and the rotation group $SO(4)$, can be regarded as the group of real fifth-order matrices having the form

$$g = \begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix} \tag{10}$$

where 0 is a row with four zero elements, while a is a four-dimensional column vector. It contains the rotations $k \in SO(4)$ and the translations $a \in T_4$. The unitary irreducible representations (UIRs) of $ISO(4)$ can be derived by the induced representation method introduced in [27] for analyzing the Poincare group $ISO(3, 1)$. It is known that they are labeled by the pair (ρ, ν), where $\nu = 0, 1, 2, \ldots$, while $0 < \rho < \infty$.

We want to study scattering problems related to the UIRs of $ISO(4)$ with $\nu = 0$, in the sense that relation (2) hold. The Schrödinger equation for such systems is essentially a condition imposed on the carrier space to be irreducible. Hence in order to find the Hamiltonians for the systems under considerations we should look for a reducible representation of $ISO(4)$ containing the $(\rho, 0)$ representations. To this end, let us consider a quasi-regular representation $T(g)$ of $ISO(4)$ realized in the Hilbert space of the square-integrable functions $f(x)$ on E^4. Generally, one can use for the construction of the quasi-regular representation the carrier space $L^2(E^4, \mathrm{d}\mu)$ with any quasi-invariant measure $\mathrm{d}\mu(x)$ on E^4. They are given by [26]

$$T(g)f(x) = (\mathrm{d}\mu(g^{-1}x)/\mathrm{d}\mu(x))^{1/2} f(g^{-1}x), \tag{11}$$

with inner product

$$(f, f') = \int \overline{f(x)} f'(x)\,\mathrm{d}\mu(x). \tag{12}$$

We can, without loss of generality, put

$$\mathrm{d}\mu(x) = h(x)\,\mathrm{d}x, \tag{13}$$

where $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4$. The requirement that the measure is quasi-invariant implies only the condition

$$h(x) \geqslant 0.$$

Since $\mathrm{d}x$ is an invariant measure on E^4, then formula (11) can be written in the form

$$T(g)f(x) = (h(g^{-1}x)/h(x))^{1/2} f(g^{-1}x). \tag{14}$$

We denote by $\{g_{ij}(t)\}, i < j, i, j = 1, 2, 3, 4$, the one-parameter subgroups of $ISO(4)$ consisting of rotations in the i–j-planes, that is, of transformations of the form

$$x'_k = x_k, k \neq i, j, \qquad x'_i = x_i \cos t + x_j \sin t, \qquad x'_j = -x_i \sin t + x_j \cos t, \tag{15}$$

while by $\{g_i(t)\}$, $i = 1, 2, 3, 4$, the one-parameter subgroups of $ISO(4)$ consisting of translations along the i-axes, that is, of transformations of the form

$$x'_k = x_k, \quad k \neq i, \qquad x'_i = x_i + t. \tag{16}$$

As is well known [25], given a representation of a group, one can always obtain infinitesimal operators via the one-parameter subgroups. Performing this for the representation (14), we find

$$J_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} + \frac{1}{2h}\left(x_i \frac{\partial h}{\partial x_j} - x_j \frac{\partial h}{\partial x_i}\right) \tag{17}$$

$$= h^{-1/2}(x) \circ \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right) \circ h^{1/2}(x) \tag{18}$$

for the one-parameter subgroups $\{g_{ij}(t)\}$ and

$$I_i = -\frac{\partial}{\partial x_i} - \frac{1}{2h}\frac{\partial h}{\partial x_i} \tag{19}$$

$$= h^{-1/2}(x) \circ \left(-\frac{\partial}{\partial x_i}\right) \circ h^{1/2}(x), \tag{20}$$

for the one-parameter subgroups $\{g_i(t)\}$, where $\circ$ denotes the composition of operators. If we compute the quadratic Casimir operator C,

$$C = \sum_{i=1}^{4} I_i^2, \tag{21}$$

it becomes

$$C = h^{-1/2}(x) \circ \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}\right) \circ h^{1/2}(x). \tag{22}$$

At this stage we note that the group $ISO(4)$ admits two Casimir operators (quadratic and quartic). It turns out that the quartic Casimir operator (i.e., the Pauli–Lubanski-type invariant) is zero for the present representation.

The quasi-regular representation $T(g)$ of the group $ISO(4)$ is decomposed onto the direct integral of $(\rho, 0)$ representations of $ISO(4)$ [25]. Hence, the $(\rho, 0)$ representation of $ISO(4)$ can be realized as a subrepresentation of the quasi-regular representation $T(g)$. Such realization is obtained if all functions f are eigenfunctions of the Casimir operator (22),

$$Cf_\rho = -\rho^2 f_\rho. \tag{23}$$

In other words, we consider the $(\rho, 0)$ representation of $ISO(4)$ realized on the eigenfunctions of the Casimir operator given by (22).

We are now in a position to extract potentials from the Casimir operator.

(i) *The* $ISO(4) \supset SO(4) \supset SO(3) \supset SO(2)$ *reduction*. Therefore, the basis functions are obtained as the common set of eigenfunctions of the Casimir operators of the subgroups in the considered reduction, i.e.

$$C^{SO(4)} f^{(1)}_{\rho lsn} = -l(l+2) f^{(1)}_{\rho lsn} \tag{24}$$

$$C^{SO(3)} f^{(1)}_{\rho lsn} = -s(s+1) f^{(1)}_{\rho lsn} \tag{25}$$

$$J_{12}^2 f^{(1)}_{\rho lsn} = -n^2 f^{(1)}_{\rho lsn}, \tag{26}$$

where $C^{SO(4)}$ and $C^{SO(3)}$ are the Casimir operators of $SO(4)$ and $SO(3)$, respectively,

$$C^{SO(4)} = \frac{1}{2}\sum_{i,j=1}^{4} J_{ij}^2, \qquad C^{SO(3)} = \frac{1}{2}\sum_{i,j=1}^{3} J_{ij}^2 \tag{27}$$

It is easy to see that

$$C^{SO(4)} = h^{-1/2}(x) \circ \left[x^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} \right) - \Lambda(\Lambda+2) \right] \circ h^{1/2}(x)$$

$$C^{SO(3)} = h^{-1/2}(x) \circ \left[\mathbf{x}^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) - L(L+1) \right] \circ h^{1/2}(x)$$

where we define $x^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, $\mathbf{x}^2 = x_1^2 + x_2^2 + x_3^2$ and

$$\Lambda = \sum_{\imath=1}^{4} x_i \frac{\partial}{\partial x_i}, \qquad L = \sum_{\imath=1}^{3} x_i \frac{\partial}{\partial x_i}.$$

Since $C^{SO(4)}$, $C^{SO(3)}$ and J_{12}^2 are sought to be diagonal, we introduce in place of x_1, x_2, x_3, x_4 the variables $r, \theta, \varphi, \beta$ via

$$x_1 = r \sin\theta \sin\varphi \sin\beta \qquad x_2 = r \sin\theta \sin\varphi \cos\beta$$
$$x_3 = r \sin\theta \cos\varphi \qquad x_4 = r \cos\theta$$

with $0 \leqslant \theta, \varphi < \pi, 0 \leqslant \beta < 2\pi$. In these coordinates the invariant measure on E^4 is given by

$$\mathrm{d}x = r^3 \sin^2\theta \sin\varphi \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\varphi \,\mathrm{d}\beta. \tag{28}$$

It is clear that we must construct the representation on the Hilbert space $\mathfrak{H}$ with the measure $\mathrm{d}\mu = r^2 \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\varphi \,\mathrm{d}\beta$, where β is an auxiliary variable. To this end, we can make use of the freedom of choosing $h(x)$. Equations (13) and (28) tell us that we have to set $h(x)$ as

$$h(x) = \left(x_1^2 + x_2^2\right)^{-1/2}. \tag{29}$$

With this choice one has

$$C^{SO(4)} = \left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right) + \frac{1}{\sin^2\theta \sin^2\varphi}\left(\frac{1}{4} + \frac{\partial^2}{\partial\beta^2} \right) + \frac{3}{4}$$

$$C^{SO(3)} = \frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin^2\varphi}\left(\frac{1}{4} + \frac{\partial^2}{\partial\beta^2} \right) + \frac{1}{4}$$

$$J_{12}^2 = \frac{\partial^2}{\partial\beta^2}$$

and

$$C = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right) + \frac{1}{r^2\sin^2\theta\sin^2\varphi}\left(\frac{1}{4} + \frac{\partial^2}{\partial\beta^2} \right). \tag{30}$$

Let $\mathfrak{H}^{(1)}$ be a subspace spanned by $f_{\rho l s n}^{(1)}$ with fixed n. Then the operator C restricted to $\mathfrak{H}^{(1)}$ becomes a differential operator in r, θ, φ; it is found that

$$C|_{\mathfrak{H}^{(1)}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right) + \frac{\frac{1}{4} - n^2}{r^2\sin^2\theta\sin^2\varphi}. \tag{31}$$

Hence the Hamiltonian

$$H^{(1)} = -\frac{1}{2}\nabla^2 + \frac{n^2 - \frac{1}{4}}{2r^2\sin^2\theta\sin^2\varphi}, \qquad n = 0, 1, 2, \ldots, \tag{32}$$

is related to $ISO(4)$ in the sense that the following relation holds:

$$H^{(1)} = -\tfrac{1}{2}C\big|_{\mathfrak{H}^{(1)}}. \tag{33}$$

That means that the $(\rho, 0)$ representation of $ISO(4)$ allows us to describe fixed energy eigenstates of a family of Hamiltonians (32) with different values of the potential parameters.

Due to the extra integral of motions

$$\widetilde{L^2} = \mathbf{L}^2 + \frac{n^2 - \frac{1}{4}}{\sin^2\theta \sin^2\varphi} \tag{34}$$

and

$$\widetilde{L_z^2} = L_z^2 + \frac{n^2 - \frac{1}{4}}{\sin^2\varphi}, \tag{35}$$

where

$$\mathbf{L}^2 = -\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right), \qquad L_z^2 = -\frac{\partial^2}{\partial\varphi^2}, \tag{36}$$

Hamiltonian (32) is separable in the spherical coordinate system. Moreover, it is not difficult to see that $\widetilde{L^2}$ and $\widetilde{L_z^2}$ are related to $C^{SO(4)}$ and $C^{SO(3)}$ in the sense that

$$\widetilde{L^2} = -\left(C^{SO(4)} - \tfrac{3}{4}\right)\big|_{\mathfrak{H}^{(1)}}, \qquad \widetilde{L_z^2} = -\left(C^{SO(3)} - \tfrac{1}{4}\right)\big|_{\mathfrak{H}^{(1)}}. \tag{37}$$

It is worth noticing that there is a simple connection between the dynamics in the three dimensions governed by (32) and a free motion in four dimensions:

$$H^{(1)} = (r\sin\theta\sin\varphi)^{1/2} \circ H^{(0)} \circ (r\sin\theta\sin\varphi)^{-1/2}|_{\mathfrak{H}^{(1)}}$$

$$\psi^{(0)}(r, \theta, \varphi, \beta) = \frac{1}{\sqrt{2\pi}}(r\sin\theta\sin\varphi)^{1/2}\psi^{(1)}(r, \theta, \varphi)\exp(in\beta)$$

where $H^{(0)}$ and $\psi^{(0)}$ are a free Hamiltonian in four dimensions and its eigenfunctions, respectively. It is clear from this remark that the potentials in (32) will be of necessity transparent. Finally, we give for reference the expression for the wavefunctions $\psi^{(1)}$,

$$\psi^{(1)}(r, \theta, \varphi) = \mathcal{R}_{kl}(r)\mathcal{Y}^{(1)}_{lsn}(\theta, \varphi), \tag{38}$$

where $\mathcal{R}_{kl}(r)$ is the radial part of the wavefunctions, while $\mathcal{Y}^{(1)}_{lsn}(\theta, \varphi)$ is the angular part of it:

$$\mathcal{R}_{kl}(r) = \sqrt{k/r}\,J_{l+1}(kr), \tag{39}$$

$$\mathcal{Y}^{(1)}_{lsn}(\theta, \varphi) = \chi^{(1)}\sin^{s+\frac{1}{2}}\theta\sin^{n+\frac{1}{2}}\varphi\, C^{1+s}_{l-s}(\cos\theta)C^{\frac{1}{2}+n}_{s-n}(\cos\varphi), \tag{40}$$

$$\chi^{(1)} = \left[\frac{2^{2s+2n}(l-s)!(s-n)!(s!)^2\Gamma^2\left(\frac{1}{2}+n\right)(1+l)(1+2s)}{(l+s+1)!(s+n)!}\right]^{\frac{1}{2}} \tag{41}$$

with $k = \sqrt{2E}$. Here $C^{\lambda}_n(t)$ are the Gegenbauer polynomials [28].

Observe that the angle-function $\mathcal{Y}^{(1)}_{lsn}(\theta, \varphi)$ depends on the details of the dynamics. This is a result of very general properties, shared by all non-central Hamiltonians. It is also worthwhile to note that the wavefunctions $\psi^{(1)}$ are related to matrix elements of $(\rho, 0)$ representations of $ISO(4)$ in the bases corresponding to $ISO(4) \supset SO(4) \supset SO(3) \supset SO(2)$ reduction [25]. Namely, they are connected with associated spherical functions of $(\rho, 0)$ with $\rho = k$.

(ii) *The* $ISO(4) \supset SO(4) \supset SO(2) \times SO(2)$ *reduction.* Now, the reduction conditions are

$$C^{SO(4)} f^{(2)}_{\rho lmn} = -l(l+2) f^{(2)}_{\rho lmn} \tag{42}$$

$$J^2_{12} f^{(2)}_{\rho lmn} = -m^2 f^{(2)}_{\rho lmn} \tag{43}$$

$$J^2_{34} f^{(2)}_{\rho lmn} = -n^2 f^{(2)}_{\rho lmn}. \tag{44}$$

The parametrization that we see for x_1, x_2, x_3, x_4 must be such as to make $C^{SO(4)}$, J^2_{12} and J^2_{34} particularly simple. We define them as follows:

$$x_1 = r \sin\theta \sin\varphi \qquad x_2 = r \sin\theta \cos\varphi$$
$$x_3 = r \cos\theta \sin\beta \qquad x_4 = r \cos\theta \cos\beta,$$

with $0 \leqslant \theta < \pi/2$, $0 \leqslant \varphi, \beta < 2\pi$. In these coordinates the invariant measure on E^4 is given by

$$\mathrm{d}x = r^3 \sin\theta \cos\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\beta. \tag{45}$$

To get the Hilbert space $\mathfrak{H}$ with the measure $\mathrm{d}\mu(x) = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\beta$, one has to set $h(x)$ as

$$h(x) = \left(x_3^2 + x_4^2\right)^{-1/2}.$$

With this choice we have

$$C^{SO(4)} = \left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{1}{\cos^2\theta}\left(\frac{1}{4} + \frac{\partial^2}{\partial\beta^2}\right) + \frac{3}{4} \tag{46}$$

$$J^2_{12} = \frac{\partial^2}{\partial\varphi^2}, \qquad J^2_{34} = \frac{\partial^2}{\partial\beta^2} \tag{47}$$

$$C = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{1}{r^2\cos^2\theta}\left(\frac{1}{4} + \frac{\partial^2}{\partial\beta^2}\right). \tag{48}$$

It is easy to check that the restriction of C to a subspace $\mathfrak{H}^{(2)}$ spanned by $f^{(2)}_{\rho lmn}$, for given n, yields

$$C|_{\mathfrak{H}^{(2)}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{\frac{1}{4} - n^2}{r^2\cos^2\theta}. \tag{49}$$

Hence the Hamiltonian

$$H^{(2)} = -\frac{1}{2}\nabla^2 + \frac{n^2 - \frac{1}{4}}{2r^2\cos^2\theta}, \qquad n = 0, 1, 2, \ldots, \tag{50}$$

is related to $ISO(4)$ in the sense that the following relation holds:

$$H^{(2)} = -\tfrac{1}{2} C\big|_{\mathfrak{H}^{(2)}}. \tag{51}$$

In this case the operator

$$\widetilde{L^2} = \mathbf{L}^2 + \frac{n^2 - \frac{1}{4}}{\cos^2\theta} \tag{52}$$

is responsible for the separability of $H^{(2)}$ in the spherical coordinate. Moreover, it is not difficult to see that $\widetilde{L^2}$ is related to $C^{SO(4)}$:

$$\widetilde{L^2} = -\left(C^{SO(4)} - \tfrac{3}{4}\right)\big|_{\mathfrak{H}^{(2)}}. \tag{53}$$

The second integral of motion is, of course, $L_z^2 = -(\partial^2/\partial\varphi^2)$ (due to azimuthal symmetry).

Finally, we give for reference the expression for the scattering wavefunctions

$$\psi^{(2)}(r,\theta,\varphi) = \mathcal{R}_{kl}(r)\mathcal{Y}^{(2)}_{lmn}(\theta,\varphi), \tag{54}$$

where $\mathcal{R}_{kl}(r)$ is given by (39), while the angle-function $\mathcal{Y}^{(2)}_{lmn}(\theta,\varphi)$ is given by

$$\mathcal{Y}^{(2)}_{lmn}(\theta,\varphi) = \chi^{(2)}\sin^m\theta\cos^{n+\frac{1}{2}}\theta\mathcal{P}^{(m,n)}_{\lambda}(\cos 2\theta)\exp(-\mathrm{i}m\varphi), \tag{55}$$

with $2\lambda = l-m-n$. Here $\mathcal{P}^{(\alpha,\beta)}_n(t)$ are Jacobi polynomials [28]. The normalization constant $\chi^{(2)}$ is given by

$$\chi^{(2)} = \left[\frac{(m+n+\lambda)!\lambda!(2\lambda+m+n+1)}{\pi(m+\lambda)!(n+\lambda)!}\right]^{\frac{1}{2}}. \tag{56}$$

It is also worth noting that the functions $\psi^{(2)}(x)$ are related to matrix elements of $(\rho,0)$ representations of $ISO(4)$ in the bases corresponding to $ISO(4) \supset SO(4) \supset SO(2)\times SO(2)$ reduction [25].

At the end, we remark that potentials (7) and (8) belong to a class of superintegrable potentials [29]. The scattering problems for these potentials have been investigated by the path integral method in [30]. Those authors do not discuss the transparency phenomenon. However, from equations (19) and (21) of [30] one may conclude that the potentials under consideration are transparent. For example, putting $\alpha=\gamma=0$ and $\beta=(4n^2-1)/8$ in (1) of [30] we come to the potential in (8). In this case the S-matrix (see equation (19) of [30]) is reduced to

$$\begin{aligned}S(\mathbf{k}'',\mathbf{k}') = (k'k'')^{-1}\delta(k''-k')\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}C_{lm}(\sin\vartheta'')^{1+m}(\cos\vartheta'')^{n+\frac{1}{2}}P_l^{(1+m,n)}(\cos 2\vartheta'')\\ \times\sin\left(\frac{\phi''}{2}\right)\cos\left(\frac{\phi''}{2}\right)P_m^{(\frac{1}{2},\frac{1}{2})}(\cos\phi'')(\sin\vartheta')^{1+m}(\cos\vartheta')^{n+\frac{1}{2}}P_l^{(1+m,n)}(\cos 2\vartheta')\\ \times\sin\left(\frac{\phi'}{2}\right)\cos\left(\frac{\phi'}{2}\right)P_m^{(\frac{1}{2},\frac{1}{2})}(\cos\phi')\end{aligned}$$

with

$$C_{lm} = \frac{4(n+m+2l+2)l!(n+m+l+1)!\Gamma^2(m+2)}{(n+l)!(m+l+1)!\Gamma^2(m+3/2)}.$$

Taking into account (21) of [30], we obtain

$$\begin{aligned}S(\mathbf{k}'',\mathbf{k}') &= (k'k'')^{-1}\delta(k''-k')\delta(\cos\vartheta''-\cos\vartheta')\delta(\phi''-\phi')\\ &= \delta(\mathbf{k}''-\mathbf{k}').\end{aligned}$$

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